

## Lecture: Conditional Expectation

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Discounted Cash Flow, Section 1.2



# Outline

## 1.2 Conditional expectation

1.2.1 Uncertainty and information

1.2.2 Rules

1.2.3 Application of the rules





Uncertainty is a distinguished feature of valuation usually modelled as different future **states of nature**  $\omega$  with corresponding cash flows  $\widetilde{FCF}_t(\omega)$ .

But: to the best of our knowledge particular states of nature play no role in the valuation equations of firms, instead one **uses expectations**  $E[\widetilde{FCF}_t]$  of cash flows.



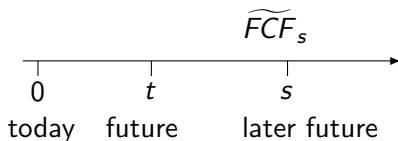


Fortune-teller

Today is certain, the future is uncertain.

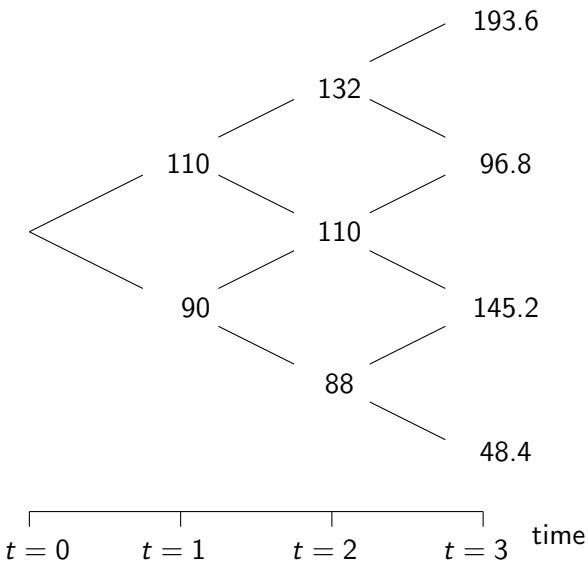
Now: **we always stay at time 0!**

And think about the future



# A finite example

3



There are three points in time in the future.

Different cash flow realizations can be observed.

The movements **up** and **down** along the path occur with probability 0.5.





Nostradamus (1503–1566),

failed fortune-teller

What happens if actual cash flow at time  $t = 1$  is neither 90 nor 110 (for example, 100)?

Our model proved to be wrong!





A.N. Kolmogorov (1903–1987),  
founded theory of  
conditional expectation

Let us think about cash flow paid at time  $t = 3$ , i.e.  $\widetilde{FCF}_3$ . What will its **expectation be tomorrow?**

This depends on the state we will have tomorrow. Two cases are possible:  
 $\widetilde{FCF}_1 = 110$  or  $\widetilde{FCF}_1 = 90$ .



**Case 1** ( $\widetilde{FCF}_1 = 110$ )

$$\implies \text{Expectation of } \widetilde{FCF}_3 = \frac{1}{4} \times 193.6 + \frac{2}{4} \times 96.8 + \frac{1}{4} \times 145.2 = 133.1.$$

**Case 2** ( $\widetilde{FCF}_1 = 90$ )

$$\implies \text{Expectation of } \widetilde{FCF}_3 = \frac{1}{4} \times 96.8 + \frac{2}{4} \times 145.2 + \frac{1}{4} \times 48.4 = 108.9.$$

Hence, expectation of  $\widetilde{FCF}_3$  is

$$E \left[ \widetilde{FCF}_3 | \mathcal{F}_1 \right] = \begin{cases} 133.1 & \text{if the development at } t = 1 \text{ is up,} \\ 108.9 & \text{if the development at } t = 1 \text{ is down.} \end{cases}$$

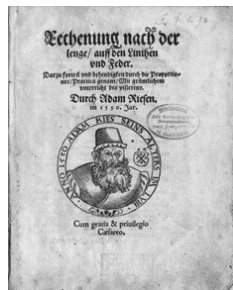


The expectation of  $\widetilde{FCF}_3$  depends on the state of nature at time  $t = 1$ . Hence, the expectation is **conditional**: conditional on the information at time  $t = 1$  (abbreviated as  $|\mathcal{F}_1$ ).

A conditional expectation covers our ideas about future thoughts.

This conditional expectation **can be uncertain**.





Arithmetic textbook of  
Adam Ries (1492–1559)

How to use conditional expectations? We will not present proofs, but only **rules for calculation**.

The first three rules will be well-known from classical expectations, two will be new.



$$E \left[ \tilde{X} | \mathcal{F}_0 \right] = E \left[ \tilde{X} \right]$$

At  $t = 0$  conditional expectation and classical expectation coincide.

Or: **conditional expectation generalizes classical expectation.**



$$E \left[ a\tilde{X} + b\tilde{Y} | \mathcal{F}_t \right] = aE \left[ \tilde{X} | \mathcal{F}_t \right] + bE \left[ \tilde{Y} | \mathcal{F}_t \right]$$

Business as usual ...



$$\boxed{E[1|\mathcal{F}_t] = 1}$$

Safety first...

From this and linearity for certain quantities  $X$ ,

$$\begin{aligned} E[X|\mathcal{F}_t] &= E[X1|\mathcal{F}_t] \\ &= X E[1|\mathcal{F}_t] \\ &= X \end{aligned}$$



Let  $s \geq t$  then

$$\boxed{E \left[ E \left[ \tilde{X} | \mathcal{F}_s \right] | \mathcal{F}_t \right] = E \left[ \tilde{X} | \mathcal{F}_t \right]}$$

When we think today about what we will know tomorrow about the day after tomorrow,

we will only know what we today already believe to know tomorrow.



If  $\widetilde{X}_t$  is known at time  $t$

$$\boxed{E \left[ \widetilde{X}_t \widetilde{Y} | \mathcal{F}_t \right] = \widetilde{X}_t E \left[ \widetilde{Y} | \mathcal{F}_t \right]}$$

We can take out from the expectation what is known.

Or: **known quantities are like certain quantities.**



We want to check our rules by looking at the finite example and an infinite example. We start with the finite example:

Remember that we had

$$E \left[ \widetilde{FCF}_3 | \mathcal{F}_1 \right] = \begin{cases} 133.1 & \text{if up at time } t = 1, \\ 108.9 & \text{if down at time } t = 1. \end{cases}$$

From this we get

$$E \left[ E \left[ \widetilde{FCF}_3 | \mathcal{F}_1 \right] \right] = \frac{1}{2} \times 133.1 + \frac{1}{2} \times 108.9 = 121.$$



And indeed

$$\begin{aligned} E \left[ E \left[ \widetilde{FCF}_3 | \mathcal{F}_1 \right] \right] &= E \left[ E \left[ \widetilde{FCF}_3 | \mathcal{F}_1 \right] | \mathcal{F}_0 \right] && \text{by rule 1} \\ &= E \left[ \widetilde{FCF}_3 | \mathcal{F}_0 \right] && \text{by rule 4} \\ &= E \left[ \widetilde{FCF}_3 \right] && \text{by rule 1} \\ &= 121 ! \end{aligned}$$

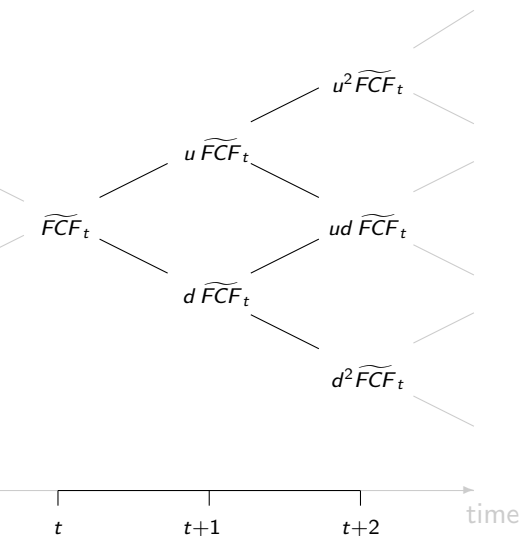


It seems **purely by chance** that

$$E \left[ \widetilde{FCF}_3 | \mathcal{F}_1 \right] = 1.1^{3-1} \times \widetilde{FCF}_1,$$

but it is on purpose! This will become clear later (when discussing autoregressive cash flows).





Again two factors up and down with probability  $p_u$  and  $p_d$  and  $0 < d < u$

or

$$\widetilde{FCF}_{t+1} = \begin{cases} u \widetilde{FCF}_t & \text{up,} \\ d \widetilde{FCF}_t & \text{down.} \end{cases}$$



Let us evaluate the conditional expectation

$$\begin{aligned} E \left[ \widetilde{FCF}_{t+1} | \mathcal{F}_t \right] &= p_u u \widetilde{FCF}_t + p_d d \widetilde{FCF}_t \\ &= \underbrace{(p_u u + p_d d)}_{:=1+g} \widetilde{FCF}_t, \end{aligned}$$

where  $g$  is the expected growth rate.

If  $g = 0$  it is said that the cash flows 'form a martingal'. In the infinite example we will later assume no growth ( $g = 0$ ).



This can be extended if  $s > t$

$$\begin{aligned}
 E \left[ \widetilde{FCF}_s | \mathcal{F}_t \right] &= E \left[ E \left[ \widetilde{FCF}_s | \mathcal{F}_{s-1} \right] | \mathcal{F}_t \right] && \text{by rule 4} \\
 &= E \left[ (1 + g) \widetilde{FCF}_{s-1} | \mathcal{F}_t \right] && \text{see above} \\
 &= (1 + g) E \left[ \widetilde{FCF}_{s-1} | \mathcal{F}_t \right] && \text{by rule 2} \\
 &= (1 + g)^{s-t} E \left[ \widetilde{FCF}_t | \mathcal{F}_t \right] && \text{repeating argument} \\
 &= (1 + g)^{s-t} \widetilde{FCF}_t && \text{by rule 5 and rule 3}
 \end{aligned}$$



We always stay in the present. Conditional expectation handles our knowledge of the future.

Five rules cover the necessary mathematics.

